Midterm: Solutions

October 17, 2025

Question 1: Olley–Pakes with Observable Characteristics (29 marks)

We are given the production equation (all logged)

$$y_{it} = \alpha_L \ell_{it} + \alpha_K k_{it} + \lambda X_{it} + \omega_{it} + e_{it} \tag{1}$$
$$\ell_{it} = f_L(k_{it}, \omega_{it}, X_{it}, r_{it}) \tag{2}$$
$$i_{it} = f_K(k_{it}, \omega_{it}, X_{it}, r_{it}) \tag{3}$$

and assumptions OP-1–OP-4 including invertibility of the investment policy in ω_{it} and the law of motion for ω_{it}:

$$\omega_{it} = E[\omega_{it} \mid \omega_{it-1}, X_{it}] + \xi_{it}.$$

We impose that the approximating functions we use are linear in their arguments (so polynomials of low degree reduce to linear forms in derivations).

(a) First-stage regression (6 marks)

Control function inversion. Under OP-1 (invertibility) we can invert the investment (proxy) function to write (assume $\phi \equiv f_K^{-1}$).

$$\omega_{it} = \phi_t(k_{it},\, i_{it},\, X_{it}) \approx \gamma_1 k_{it} + \gamma_2 i_{it} + \gamma_3 X_{it} + \gamma_t.$$

Substitute this into the production equation (1):

$$\begin{aligned}
y_{it} &= \alpha_L \ell_{it} + \alpha_K k_{it} + \lambda X_{it} + \omega_{it} + e_{it} && (1)\\
&= \alpha_L \ell_{it} + \alpha_K k_{it} + \lambda X_{it} + \gamma_1 k_{it} + \gamma_2 i_{it} + \gamma_3 X_{it} + \gamma_t + e_{it} && (2)\\
&= \alpha_L \ell_{it} + \underbrace{(\alpha_K + \gamma_1)\, k_{it} + \gamma_2 i_{it} + (\lambda + \gamma_3)\, X_{it} + \gamma_t}_{\varphi_t} + e_{it} && (3)\\
&= \alpha_L \ell_{it} + \underbrace{\gamma_1' k_{it} + \gamma_2' i_{it} + \gamma_3' X_{it} + \gamma_t'}_{\varphi_t} + e_{it} && (4)
\end{aligned}$$

(b) Second-stage regression (6 marks)

Define $\hat{\varphi}_{it}$ as followed

$$\hat{\varphi}_{it} = \gamma_1' k_{it} + \gamma_2' i_{it} + \gamma_3' X_{it} + \gamma_t' \tag{5}$$

Note that $\omega_{it} = \hat{\varphi}_{it} - \alpha_K \cdot k_{it} - \lambda X_{it}$ with OP-3 gives us

$$\begin{aligned}
\hat{\varphi}_{it} &= \alpha_K k_{it} + \lambda X_{it} + \pi_1 (\hat{\varphi}_{it-1} - \alpha_K \cdot k_{it-1} - \lambda X_{it-1}) + \pi_2 X_{it} + \xi_{it} && (6)\\
\Rightarrow \hat{\varphi}_{it} &= \alpha_K k_{it} + (\lambda + \pi_2) X_{it} + \pi_1 \hat{\varphi}_{it-1} - \pi_1 \alpha_K k_{it-1} - \pi_1 \lambda X_{it-1} + \xi_{it} && (7)
\end{aligned}$$

(c) Which parameters are identified? (3 marks)

All three are identified.

α_K is identified as coefficient of k_{it} in second stage. α_L is identified as coefficient of l_{it} in first stage. λ is identified as coefficient of X_{it-1} divided by π_1 in second stage, (where π_1 is coefficient of $\hat{\varphi}_{it-1}$ in second stage.

R code: implementation steps (2+4+2+2+4 = 14 marks total for coding subsections). Below is R / tidyverse + fixest-style code following the instructions. It assumes the data frame is named `df` and is already sorted by firmid and year (panel format). We use $\delta = 0.9$.

(2)(a) Create investment proxy i_{it} (2 marks)

```
df <- df %>%
  mutate(i_it = lead(k) - (1 -  0.9) *k)
```

(2)(b) First-stage regression approximating φ_t using 3rd degree polynomial (4 marks)

```
first_stage <- feols(y ~ l + I(year):poly(k,i_it,X,degree=3), data = df)
```

(2)(c) Create $\widehat{\varphi}_{it}$ (2 marks)

```
df$phi_hat <- fitted(first_stage)-coef(first_stage)["l"]*df$l
```

(2)(d) Create lagged variables needed for second stage (2 marks)

We need lagged k and $\hat{\varphi}$. Create lags:

```
df <- df %>%
mutate(k_lag = lag(k,1),
        X_lag = lag(X,1),
        phi_hat_lag = lag(phi_hat,1))
```

(2)(e) Second-stage regression approximating h using 3rd degree polynomial (4 marks)

```
second_stage <- feols(phi_hat ~ k + poly(phi_hat_lag,k_lag,X_lag,X,degree=3), data = df)
```

Question 2: Interpreting Results (22 marks)

We are given the estimated table (reproduced succinctly):

	OLS-1	OLS-2	AB
Quality, α_q	—	0.8 (0.02)	—
Capital, α_k	0.46 (0.02)	0.28 (0.05)	0.30 (0.04)
Labor, α_ℓ	0.67 (0.015)	0.20 (0.04)	0.25 (0.03)
R^2	0.870	0.983	0.998
Wald test (p-val)	–	–	$< 2.2 \times 10^{-16}$
Sargan (p-val)	–	–	1.00
AR(1) (p-val)	–	–	0.197
AR(2) (p-val)	–	–	0.735

(1) Covariance of inputs and q explaining changes from OLS-1 to OLS-2 (3 marks)

When quality q is omitted (OLS-1), its influence on output is absorbed by observed inputs L and K if q is positively correlated with these inputs. OLS-1 shows large elasticities for both labor (0.67) and capital (0.46). After including observed quality in OLS-2 (with $\alpha_q = 0.8$), much of the variation previously attributed to inputs is re-allocated to quality: α_k falls to 0.28 and α_ℓ falls sharply to 0.20. This pattern is consistent with q being positively correlated with both inputs: when q is included, it "soaks up" part of the returns formerly attributed to L and K.

(2) Are OLS-2 (or OLS-1 due to my typo) and AB estimates different? (4 marks)

OLS-2

Compute differences and test using the joint standard error approximation given:

$$\text{joint stderr} = \sqrt{\text{stderr}(\alpha^{\text{OLS-2}})^2 + \text{stderr}(\alpha^{\text{AB}})^2}.$$

Capital: $\alpha_k^{\text{OLS-2}} = 0.28$ (0.05), $\alpha_k^{\text{AB}} = 0.30$ (0.04).

$$\text{diff} = 0.28 - 0.30 = -0.02.$$

Joint stderr $= \sqrt{0.05^2 + 0.04^2} \approx 0.0640$. Test statistic $t = -0.02/0.0640 \approx -0.312$. $|t| < 1.96$, so we *cannot* reject equality at 5%: the two estimates are not statistically different.

Labor: $\alpha_\ell^{\text{OLS-2}} = 0.20$ (0.04), $\alpha_\ell^{\text{AB}} = 0.25$ (0.03).

$$\text{diff} = 0.20 - 0.25 = -0.05.$$

Joint stderr $= \sqrt{0.04^2 + 0.03^2} = 0.05$. Test statistic $t = -0.05/0.05 = -1.0$. Again $|t| < 1.96$, so we *cannot* reject equality at 5%. Thus OLS-2 and AB estimates are *not* significantly different for these inputs.

OLS-1

We now test whether the *OLS-1* and *AB* estimates differ significantly for both capital and labor. Following the question, we use the joint standard error formula:

$$\text{Joint Std. Err.} = \sqrt{\text{stderr}(\hat{\alpha}^{\text{OLS-1}})^2 + \text{stderr}(\hat{\alpha}^{\text{AB}})^2}.$$

Capital.

$$\hat{\alpha}_k^{\text{OLS-1}} = 0.46\ (0.02), \qquad \hat{\alpha}_k^{\text{AB}} = 0.30\ (0.04).$$

$$\text{Difference} = 0.46 - 0.30 = 0.16, \qquad \text{Joint Std. Err.} = \sqrt{0.02^2 + 0.04^2} = 0.0447.$$

Hence

$$t = \frac{0.16}{0.0447} \approx 3.58.$$

Because $|t| > 1.96$, the null of equality is rejected at the 5% level: the capital elasticity is significantly higher in OLS-1 than in AB.

Labor.

$$\hat{\alpha}_\ell^{\text{OLS-1}} = 0.67\ (0.015), \qquad \hat{\alpha}_\ell^{\text{AB}} = 0.25\ (0.03).$$

$$\text{Difference} = 0.67 - 0.25 = 0.42, \qquad \text{Joint Std. Err.} = \sqrt{0.015^2 + 0.03^2} = 0.0335.$$

Thus

$$t = \frac{0.42}{0.0335} \approx 12.54,$$

again far exceeding the critical value. We therefore strongly reject equality: the OLS-1 labor elasticity is much larger than the AB estimate.

(3) Are the instruments in AB exogenous? Are unobservables serially correlated? (3 marks)

Instrument exogeneity: The Sargan test p-value is 1.00 (very large), so we fail to reject that the instruments are exogeneous; this is consistent with the instruments being exogenous (not evidence against exogeneity).

Serial correlation: AR(1) p-value = 0.197 and AR(2) p-value = 0.735. The AR(2) test in difference-GMM is the key test for serial correlation in the *original* errors; a non-rejection (p=0.735) indicates no evidence of second-order serial correlation, which supports the validity of instruments constructed from lagged levels. AR(1) p=0.197 also fails to reject AR(1) (though AR(1) is usually expected in differences). Overall we do not have evidence of problematic serial correlation of unobservables that would invalidate the instruments.

(4) Could q be a control function for l and k? (2 marks)

If says yes: Yes. A control function is an observed variable that, when conditioned on, removes endogeneity between inputs and the unobserved productivity term. Here, if q is observed and captures the component of productivity ω_{it} that is correlated with inputs L and K, then including q in the regression controls for that source of endogeneity. Thus q functions as a control variable (control function) for L and K, provided its inclusion indeed renders (L, K) exogenous conditional on q. This is further supported by the fact that the OLS of log output over quality, log capital, and log labour give same elasticity estimates as that of AB.

If says no: No. A control function is an observed variable that, when conditioned on, removes endogeneity between inputs and the unobserved productivity term. However, clearly in the model even if q_{it} is correlated with ω_{it}, managers still make optimal labour and capital decisions based of on ω_{it}. This implies that controlling for quality does not provide a residual which is independent of labor and capital.

(5) Are $K = 0.5$ and $L = 200$ under- or over-utilized? (10 marks)

We are told: $A(q_{it}) = 1$, prices $W_L = W_K = 1$. Manager's objective: maximize

$$\Pi = Y - W_L L - W_K K = L^{\alpha_\ell} K^{\alpha_k} - L - K,$$

and we are instructed to use AB estimates: $\alpha_\ell = 0.25$, $\alpha_k = 0.30$.

Compute output at given inputs.

$$Y = L^{\alpha_\ell} K^{\alpha_k} = 200^{0.25} \cdot 0.5^{0.30}.$$

This gives $Y = 3.05$.

Marginal products and comparison to marginal cost (price = 1). Marginal product of labor:

$$\text{MP}_L = \frac{\partial Y}{\partial L} = \alpha_\ell \, L^{\alpha_\ell - 1} K^{\alpha_k} = \alpha_\ell \frac{Y}{L}.$$

Similarly for capital:

$$\text{MP}_K = \alpha_k \frac{Y}{K}.$$

Compute numerically:

- $\text{MP}_L = 0.25 \cdot \frac{3.05}{200} \approx 0.00381.$
- $\text{MP}_K = 0.30 \cdot \frac{3.05}{0.5} \approx 1.83$

Interpretation:

- For labor: $\text{MP}_L \approx 0.00382 \ll W_L = 1$. Since the marginal product of an extra unit of labor is far less than its cost (wage = 1), the firm is *over-utilizing* labor — i.e., it should reduce labor to increase profit.
- For capital: $\text{MP}_K \approx 1.8331 > W_K = 1$. Since the marginal product of capital exceeds its cost, capital is *under-utilized* — i.e., the firm should increase capital to raise profit.

Conclusion: At $(K = 0.5, L = 200)$, labor is over-utilized and capital is under-utilized given the AB elasticities.
Other methods to answer this question:

- Give **6 marks** if people start with cost-minimization problem (or state the cost function formula and then try to use it) instead and test 2nd order condition (even if wrong conclusions).
- Give **full marks** if people find optimal values of L and K to answer underutilized and overutilized.
- Give **6 marks** if people at least derive first order condition of the Profit maximization problem but then fail to either use a.) MPL and MPK to find answers or b) fail to find the optimal values of L and K to use them for comparison and then.