UG Empirical Industrial Organization

Lecture 2: Production Functions: Introduction

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[1] This course is based on Victor Aguirregabiria's Empirical IO book. The slides (in my first year of teaching) are extremely close to his slides. These will change in the future iterations of this course.

Outline

- **MODEL**
- **DATA**
- **WHAT DETERMINES PRODUCTIVITY?**
- **ESTIMATION: THE SIMULTANEITY PROBLEM**

What is a Production Function (PF)?

- It is a function that relates the amount of physical output of a production process (Y) to the amount of physical inputs or factors of production (X).
- **Estimation of Production Functions (PFs) plays a key role in empirical questions such as:**
  - Estimation of Firms' Costs.
  - Measurement of Firms' Productivity (and its growth).
  - Misallocation of inputs:
    - Do more productive firms use more inputs?
  - Evaluating the effects of adopting new technologies.
  - Measurement of Learning-by-doing.

Production Functions

- **A standard representation of a Production Function (PF) is:**

$$Y = A \times f(X_1, X_2, ..., X_J)$$

  - Y is the firm's output (physical units per year).
  - $X_1, X_2, ..., X_J$ are measures of J firm inputs:
    - Labor, capital, energy, materials, ...
  - A represents the firm's Total Factor Productivity (TFP).
- **TFP captures anything else affecting output that we do not observe as researchers:**
  - Managerial ability, Organization of production, Quality of inputs., Quality of land.
- **The marginal productivity of input j is:**

$$MP_j = \frac{dY}{dX_j} = A\frac{df}{dX_j}$$

- **Note:** TFP increases proportionally the marginal productivity of all inputs.

Cobb-Douglas Production Function

- A common specification is the Cobb-Douglas Production Function (PF):
$$Y = A X_1^{\alpha_1} X_2^{\alpha_2} \dots X_J^{\alpha_J}$$
- Interpretation of parameters:
  - $\alpha_1, \alpha_2, ..., \alpha_J$ are technological parameters.
  - All α_j are positive.
- Marginal productivity of input j:
$$MP_j = \frac{dY}{dX_j} = \alpha_j \cdot A \cdot X_1^{\alpha_1} \dots X_j^{\alpha_j - 1} \dots X_J^{\alpha_J} = \alpha_j \frac{Y}{X_j}$$
- Cross-effects on marginal productivity:
  - The marginal productivity of input j increases with the amount of any other input $k \neq j$.
  - This implies that all inputs are **complements** in production.
$$\frac{dMP_j}{dX_k} = \frac{\alpha_j}{X_j} \frac{dY}{dX_k} = \alpha_j \alpha_k \frac{Y}{X_j X_k} > 0$$

Production Function and Cost Function

- Given a firm's Production Function (PF) and input prices, its cost function $C(Y)$ is defined as the minimum cost of producing output Y:

$$C(Y) = \left[\begin{array}{l} \min\limits_{\{X_1, X_2, ..., X_J\}} W_1 X_1 + W_2 X_2 + ... + W_J X_J \\ \text{Subject to: } Y = Af(X_1, X_2, ..., X_J) \end{array} \right]$$

- **Where:**
  - $W_1, W_2, ..., W_J$ are input prices.
- **Lagrange Representation:**

$$C(Y) = \min_{\{\lambda, X_1, ..., X_J\}} [W_1 X_1 + ... + W_J X_J + \lambda (Y - Af(X_1, ..., X_J))]$$

  - λ is the Lagrange multiplier of the output restriction.
- Marginal conditions of optimality for every input j:

$$W_j - \lambda MP_j = 0$$

Cost Function for the Cobb-Douglas PF

- Remember that for the Cobb-Douglas Production Function:

$$MP_j = \alpha_j \frac{Y}{X_j}$$

- The marginal condition of optimality for input j implies:

$$W_j X_j = \lambda \alpha_j Y$$

- The total cost is equal to:

$$\sum_{j=1}^{J} W_j X_j = \lambda \alpha Y, \quad \text{where} \quad \alpha \equiv \sum_{j=1}^{J} \alpha_j$$

- Parameter α measures the Returns to Scale in the Production Function.
- Returns to Scale: constant if $\alpha = 1$, decreasing if $\alpha < 1$, increasing if $\alpha > 1$.

Derivation of Cost Function for Cobb-Douglas [2]

- We need to obtain the value of the Lagrange multiplier λ.
- Solving the marginal conditions $X_j = \frac{\lambda \alpha_j Y}{W_j}$ into the Production Function:

$$Y = A\left(\frac{\lambda \alpha_1 Y}{W_1}\right)^{\alpha_1}\left(\frac{\lambda \alpha_2 Y}{W_2}\right)^{\alpha_2}\cdots\left(\frac{\lambda \alpha_J Y}{W_J}\right)^{\alpha_J}$$

- Solving for the Lagrange multiplier:

$$\lambda = A^{\frac{-1}{\alpha}} \cdot Y^{\frac{1}{\alpha}-1} \cdot \prod_{j=1}^{J}\left(\frac{W_j}{\alpha_j}\right)^{\frac{\alpha_j}{\alpha}}$$

- Plugging this expression into the cost equation $C(Y) = \lambda \alpha Y$, we obtain the cost function:

$$C(Y) = \alpha \cdot A^{\frac{-1}{\alpha}} \cdot Y^{\frac{1}{\alpha}} \cdot \prod_{j=1}^{J}\left(\frac{W_j}{\alpha_j}\right)^{\frac{\alpha_j}{\alpha}}$$

Cost Function for the Cobb-Douglas PF (3)

- The cost function is given by:

$$C(Y) = \alpha \cdot A^{\frac{-1}{\alpha}} \cdot Y^{\frac{1}{\alpha}} \cdot \left(\frac{W_1}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha}} \cdot \left(\frac{W_2}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha}} \cdot \cdots \left(\frac{W_J}{\alpha_J}\right)^{\frac{\alpha_J}{\alpha}}$$

- The marginal cost is:

$$C'(Y) = Y^{\frac{1}{\alpha}-1} \left(\frac{1}{A}\right)^{\frac{1}{\alpha}} \left(\frac{W_1}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha}} \left(\frac{W_2}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha}} \cdots \left(\frac{W_J}{\alpha_J}\right)^{\frac{\alpha_J}{\alpha}}$$

- The sign of $C''(Y)$ is determined by the sign of $\frac{1}{\alpha} - 1$:
  - If $\alpha = 1$ (constant returns to scale) then $C''(Y) = 0$ (Constant Marginal Cost)
  - If $\alpha < 1$ (decreasing returns to scale)then $C''(Y) > 0$ (Increasing Marginal Cost)
  - If $\alpha > 1$ (increasing returns to scale) then $C''(Y) < 0$ (Decreasing Marginal Cost)

More on the Cobb-Douglas

- A useful property of the Cobb-Douglas Production Function (for estimation) is that its logarithmic transformation is linear in parameters:

$$\ln(Y) = \ln(A) + \alpha_1 \ln(X_1) + \alpha_2 \ln(X_2) + \cdots + \alpha_J \ln(X_J)$$

- We represent $\ln(Y)$ and $\ln(X)$ using lower-case letters y and x, respectively, and define log-TFP as ω, such that:

$$y = \omega + \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_J x_J$$

- Differences in log-TFP (ω) are in percentage terms:

  Example: Consider two firms, 1 and 2, using the same amount of inputs X but with:$\omega_1 = 1.1$, and $\omega_2 = 1.5$ such that $\omega_2 - \omega_1 = 0.4$. Therefore, firm 2 is 40% more productive than firm 1.

More on the Cobb-Douglas (2)

- Most empirical applications in this course consider two inputs: Labor (L) and Capital (K):

$$y = \alpha_L \ell + \alpha_K k + \omega$$

  - Where $\ell \equiv \ln(L)$ and $k \equiv \ln(K)$.
- Sometimes the specification also includes Materials (M):

$$y = \alpha_L \ell + \alpha_K k + \alpha_M m + \omega$$

  - Where $m \equiv \ln(M)$.

Data

- Panel data of N firms over T time periods with information on output, labor, and capital (in logs):

$$\{y_{it}, \ell_{it}, k_{it} : i = 1, 2, ..., N;\ t = 1, 2, ..., T\}$$

- We are interested in estimating the parameters α_L and α_K in the Cobb-Douglas Production Function (in logs):

$$y_{it} = \alpha_L \ell_{it} + \alpha_K k_{it} + \omega_{it} + e_{it}$$

- ω_{it} represents log-TFP, which includes unobserved inputs (for the researcher) that are known to the firm when deciding K and L, such as: Managerial ability, Quality of land, Different technologies

Observing Revenue Instead of Physical Output

- Revenue is defined as:

$$R_{it} = P_{it} Y_{it}$$

- Taking the logarithm:

$$\ln(R_{it}) = \ln(P_{it}) + \ln(Y_{it}) = p_{it} + y_{it}$$

- The researcher only observes $\ln(R_{it})$.
- **"Solution" 1:** Proxy $\ln(P_{it})$ using an industry-level price index, $\ln(P_{\text{Industry},t})$.

$$\ln(P_{it}) = \ln(P_{\text{Industry},t}) + u_{it}$$

- The measurement error u_{it} can be interpreted as part of the total factor productivity ω_{it}.

Observing Revenue Instead of Physical Output [2]

- **"Solution" 2:** Assume isoelastic demand and monopolistic competition:

$$y_{it} = b_{it} - \beta p_{it}$$

  Here, b_{it} is unobservable to the researcher, and β is the demand elasticity.

- Then $p_{it} = \frac{b_{it} - y_{it}}{\beta}$ and substituting into the revenue equation:

$$\ln(R_{it}) = p_{it} + y_{it} = \frac{b_{it}}{\beta} + \left(1 - \frac{1}{\beta}\right) y_{it}$$

- This leads to the estimation equation:

$$\ln(R_{it}) = \alpha_L^* \ell_{it} + \alpha_K^* k_{it} + \omega_{it}^* + e_{it}$$

  where $\alpha_L^* = \left(1 - \frac{1}{\beta}\right) \alpha_L$, $\alpha_K^* = \left(1 - \frac{1}{\beta}\right) \alpha_K$, $\omega_{it}^* = \omega_{it} + \frac{b_{it}}{\beta}$

- This is very relevant for the interpretation of results and of "log-TFP."

- For instance: $\alpha_L^* + \alpha_K^* = \left(1 - \frac{1}{\beta}\right)(\alpha_L + \alpha_K) < \alpha_L + \alpha_K$

The Simultaneity Problem

- Consider the Production Function:

$$y_{it} = \alpha_L \ell_{it} + \alpha_K k_{it} + \omega_{it}$$

- We are interested in estimating α_L and α_K.
- These parameters represent *ceteris paribus* causal effects of labor and capital on output, respectively.
- When the manager decides the optimal (k_{it}, ℓ_{it}), she has some information about log-TFP ω_{it} that we do not observe.
- This means there is a correlation between the observable inputs (k_{it}, ℓ_{it}) and the unobservable ω_{it}.
- This correlation implies that the OLS estimators of α_L and α_K are biased and inconsistent.